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Motions in a Bose condensate
I. The structure of the large circular vortex

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Abstract. A procedure is given which in principle permits the exact evaluation of any term in the asymptotic expansion in a/c of the velocity U_0 and energy E of a circular vortex line of radius c in a Bose condensate (where a is the healing length). The procedure is used to obtain the two leading terms (previously only inaccurately determined by a variational calculation), namely,

$$U_0 = \frac{\kappa}{4\pi c}\left\{\ln\left(\frac{8c}{a}\right) - 0{\cdot}615\right\}$$

$$E = \tfrac{1}{2}\rho\kappa^2 c\left\{\ln\left(\frac{8c}{a}\right) - 1{\cdot}615\right\}$$

where κ denotes the quantum of circulation, and ρ denotes the fluid density at infinity.

1. Introduction

It is well known that the kinetic energy $\mathscr{E}$ per unit length of a classical rectilinear vortex line of circulation κ and core radius a in a fluid of density ρ can be written in the form

$$\mathscr{E} = \mathscr{E}_{\mathrm{E}} + \mathscr{E}_{\mathrm{I}} \tag{1.1}$$

where $\mathscr{E}_{\mathrm{E}}$ is the kinetic energy of the motions in the exterior of the core and $\mathscr{E}_{\mathrm{I}}$ is that of the motions in the interior of the core:

$$\mathscr{E}_{\mathrm{E}} = \frac{\kappa^2\rho}{4\pi}\ln\left(\frac{C}{a}\right) \tag{1.2}$$

$$\mathscr{E}_{\mathrm{I}} = \pi\rho\int_0^a u_\phi{}^2 r\,\mathrm{d}r. \tag{1.3}$$

Here C is the cut-off distance, representing the size of the container which must be introduced so that $\mathscr{E}_{\mathrm{E}}$ converges, and $u_\phi(r)$ is the circular velocity within the core of the vortex and is such that $u_\phi(a) = \kappa/2\pi a$ (the circulation condition); r, ϕ, z are cylindrical coordinates. The simplest internal structure to assume is the uniform vorticity core for which $u_\phi = \kappa r/2\pi a^2$, giving $\mathscr{E}_{\mathrm{I}} = \rho\kappa^2/16\pi$.

It is physically clear that the flow associated with a circular vortex ring whose radius c is large compared with a must resemble closely the corresponding rectilinear vortex above, provided its internal structure is the same.† In fact it may be shown

† Strictly speaking, it is impossible to obtain a steady-state circular vortex by postulating a constant vorticity across its core (Lamb 1945—§ 161) but steady solutions may be obtained by assuming that the vorticity of the ring is $M\tilde{\omega}$, where $\tilde{\omega}$ denotes distance from the axis of symmetry and M is a constant. For small a/c there is scarcely any difference.

that the total (kinetic) energy E of the circular vortex is given by

$$E = 2\pi c\mathscr{E} \tag{1.4}$$

(i.e. $\mathscr{E}$ times its circumference) provided the cut-off distance C in (1.2) is properly related to c. Indeed since, to the leading order (as $a/c \to 0$),

$$\mathscr{E}_{\mathrm{E}} = \frac{\rho\kappa^2}{4\pi}\left\{\ln\left(\frac{8c}{a}\right)-2\right\} \tag{1.5}$$

we should take $C = 8c/e^2$, so that, for example, for a vortex with a uniform vorticity core, we have simply

$$E = \tfrac{1}{2}\rho\kappa^2 c\left\{\ln\left(\frac{8c}{a}\right)-\frac{7}{4}\right\}. \tag{1.6}$$

A simple, clear-cut, qualitatively correct model of vortices in liquid helium is provided by the Bose condensate which can be characterized by a wave function ψ whose normalization yields the total number of condensed particles (Bogoliubov 1947, Penrose and Onsager 1956). This low-density approximation evades the quantum many-body-problem by a self-consistent field approximation, incorporating a short-range repulsive potential V_0 of delta-function type, and was proposed in 1958 by Ginzberg and Pitaevskii (1958—see also Gross 1961 and 1963). The system is then described by the one-particle distribution wave function $\psi(\boldsymbol{x}, t)$ obeying

$$\mathrm{i}\hbar\frac{\partial\psi}{\partial t} = -\frac{\hbar^2}{2M}\nabla^2\psi + V_0\psi|\psi|^2 \tag{1.7}$$

where M is the particle mass, and

$$\int_v |\psi|^2\,\mathrm{d}V = N \tag{1.8}$$

is the total number of particles. The number current density is

$$\boldsymbol{j} = \frac{\hbar}{2\mathrm{i}}\int_v (\psi^*\nabla\psi - \psi\nabla\psi^*)\,\mathrm{d}V. \tag{1.9}$$

The structure of the rectilinear vortex in this theory has been fully examined by Ginzburg and Pitaevskii (1958) who find results of the form (1.1) and (1.2) provided $\mathscr{E}$ is redefined by

$$\mathscr{E}_{\mathrm{I}} = \frac{\kappa^2\rho_\infty}{4\pi}\int_0^\infty \left\{\frac{\psi_0{}^2}{r}+r\left(\frac{\mathrm{d}\psi_0}{\mathrm{d}r}\right)^2+\tfrac{1}{2}(\psi_0{}^2-1)^2 r\right\}\mathrm{d}r \tag{1.10}$$

where $\psi = \psi_0\exp(\mathrm{i}\phi)$, and the bar through the integral sign denotes the finite or convergent part of the integral: its presence is required since the integral diverges logarithmically at its upper limit, and this divergence has already been recognized through the introduction of the cut-off distance C in $\mathscr{E}_{\mathrm{E}}$. Here a and ρ_∞ are the quantum core radius and density to be defined below. Ginsburg and Pitaevskii (1958) obtained, by numerical integration, the result

$$\mathscr{E} = \frac{\kappa^2\rho_\infty}{4\pi}\left\{\ln\left(\frac{C}{a}\right)+0{\cdot}38\right\} \tag{1.11}$$

(a value we have independently confirmed).

The question of the form of E for a circular vortex ring has been the subject of an intricate variational calculation by Amit and Gross (1966). It is one of the objectives of this paper to demonstrate that E may be obtained in precisely the manner expected from the discussion of the classical vortex above: (1.4) still holds provided we take $C = 8c/e^2$ in (1.11), and therefore

$$E = \tfrac{1}{2}\rho_\infty \kappa^2 c\left\{\ln\left(\frac{8c}{a}\right) - 1{\cdot}62\right\}. \tag{1.12}$$

This result is, to the order in c/a displayed, exact. That this should be the case has already been conjectured by Donnelly and Roberts (1969—§ 3). We also obtain independently from an asymptotic analysis the velocity U_0 and impulse p of the ring as

$$U_0 = \frac{\kappa}{4\pi c}\left\{\ln\left(\frac{8c}{a}\right) - 0{\cdot}62\right\} \tag{1.13}$$

$$p = \pi\rho_\infty \kappa c^2 \tag{1.14}$$

thereby verifying that the Hamiltonian relation $U_0 = \partial E/\partial p$ is satisfied. A secondary objective of this paper is to lay the foundation for the second paper in this series, in which the vibrations of vortex lines in the condensate are examined.

2. Basic theory

Equations (1.7) to (1.9) may be reduced to fluid mechanical form by writing

$$\psi = R\exp(\mathrm{i}S/\hbar) \tag{2.1}$$

where R and S are real. Substituting into (1.7) and separating real and imaginary parts, we obtain

$$\frac{\partial R^2}{\partial t} = -\nabla\cdot\left(\frac{R^2}{M}\nabla S\right) \tag{2.2}$$

and

$$-\frac{\partial S}{\partial t} = \frac{1}{2M}(\nabla S)^2 + \Pi \tag{2.3}$$

where

$$\Pi = -\frac{\hbar^2}{2M}\frac{\nabla^2 R}{R} + V_0 R^2. \tag{2.4}$$

By (1.8) and (1.9), we have

$$\int_v R^2\,\mathrm{d}V = N \tag{2.5}$$

$$\int_v R^2\,\nabla S\,\mathrm{d}V = \boldsymbol{j}. \tag{2.6}$$

Thus MR^2 is the mass density and $\nabla S/M$ may be regarded as the fluid velocity $\boldsymbol{u}$. (Then (2.6) becomes $\int_v \rho\boldsymbol{u}\,\mathrm{d}V = \boldsymbol{j}$.) Write, therefore,

$$\rho = MR^2 \tag{2.7}$$

$$\phi = -\frac{1}{M}(E_v t + S) \tag{2.8}$$

where E_v is a constant. It is clear from (2.1) that, since ψ must, on physical grounds, be single valued, the multivaluedness of S must be restricted to additive multiples of $2\pi\hbar = h$. Then (2.8) indicates that ϕ can be changed only by units of h/M; in particular, the unit of circulation for a vortex flow, namely $\kappa = \int_C \boldsymbol{u} \, . \, \mathrm{d}\boldsymbol{r}$ (= change in ϕ around a circuit C threading the core) must be h/M.

In terms of the new variables (2.7) and (2.8), equations (2.2) to (2.4) become

$$\frac{\partial \rho}{\partial t} + \nabla \, . \, (\rho \boldsymbol{u}) = 0 \tag{2.9}$$

$$\frac{\partial \phi}{\partial t} = \tfrac{1}{2}\boldsymbol{u}^2 + \Phi \tag{2.10}$$

where

$$\boldsymbol{u} = -\nabla \phi \tag{2.11}$$

and

$$\Phi = \frac{(\Pi - E_v)}{M} \tag{2.12}$$

or equivalently

$$\Phi = \frac{(V_0 R^2 - E_v)}{M} - \frac{\hbar^2}{2M^2}\frac{\nabla^2 R}{R} \tag{2.13}$$

where the last term on the right is the so-called 'quantum pressure' (cf. Gross 1963). Writing

$$a = \frac{\hbar}{(2ME_v)^{1/2}}, \qquad \rho_\infty = \frac{ME_v}{V_0} \tag{2.14}$$

(2.13) reduces to

$$\Phi = \frac{E_v}{M}\left(\frac{\rho}{\rho_\infty} - 1 - a^2 \frac{\nabla^2 \rho^{1/2}}{\rho^{1/2}}\right). \tag{2.15}$$

Here a is often called *the healing length*, and in applications is typically a few Å. For systems of linear dimensions L large compared with a, the solutions may be conveniently thought of in two parts: (i) an exterior solution, far from boundaries and vortex lines, in which the last term in (2.15) is negligible, and (ii) an interior solution having the nature of a boundary layer, here called a healing layer, for points near boundaries or near to the cores of vortex lines. The quantity L/a might be called a *quantum Reynolds number*.

The expectation value of the total energy $\mathscr{E}$ of the system is given by (Fetter 1965)

$$\mathscr{E} = \frac{\hbar^2}{2M}\int_v |\nabla\psi|^2 \, \mathrm{d}V + \frac{V_0}{2}\int_v |\psi|^4 \, \mathrm{d}V \tag{2.16}$$

where the delta-function potential has been used; alternatively, by (2.1),

$$\mathscr{E} = \frac{\hbar^2}{2M}\int_v (\nabla R)^2 \, \mathrm{d}V + \frac{1}{2M}\int_v R^2(\nabla S)^2 \, \mathrm{d}V + \frac{V_0}{2}\int_v R^4 \, \mathrm{d}V. \tag{2.17}$$

The first of these terms can be regarded as a 'quantum energy'. The second is the usual kinetic energy and the third is a potential energy term. The energy relative to the ground state is obtained by subtracting from $\mathscr{E}$ the energy of a uniform system having the same number of particles.

If we suppose that the fluid fills a box of volume $\mathscr{V}$ and that initially it is at rest at density R_u and potential energy $\mathscr{E}_u$ then

$$\mathscr{E}_u = \tfrac{1}{2}V_0 \int_{\mathscr{V}} R_u^4 \, dV = \tfrac{1}{2}V_0 R_u^4 \mathscr{V} \tag{2.18}$$

and the total number of particles is

$$N = \int_{\mathscr{V}} R_u^2 \, dV = R_u^2 \mathscr{V}. \tag{2.19}$$

In many applications, such as those we consider in this paper, it is supposed that a disturbance is created in a localized region $\mathscr{R}$ of the box far from the walls and results are sought which are valid in the limit $\mathscr{V} \to \infty$, the disturbed volume $\mathscr{R}$ being fixed. The potential energy is increased by the disturbance from (2.18) to the value

$$\mathscr{E}_v = \tfrac{1}{2}V_0 \int_{\mathscr{V}} R^4 \, dV \tag{2.20}$$

but the number of particles

$$N = \int_{\mathscr{V}} R^2 \, dV \tag{2.21}$$

will be the same. Outside $\mathscr{R}$, R will take an almost constant value R_∞, say. In the limit $\mathscr{V} \to \infty$ ($\mathscr{R}$ remaining fixed), (2.21) will give, in order of magnitude,

$$N = (\mathscr{V} - \mathscr{R})R_\infty^2 + \mathscr{R}\bar{R}^2 \tag{2.22}$$

where $\bar{R}$ denotes the mean value of R over $\mathscr{R}$. Thus by (2.19) and (2.22)

$$R_u = R_\infty + O\left(\frac{1}{\mathscr{V}}\right), \qquad \mathscr{V} \to \infty. \tag{2.23}$$

The point that R_u and R_∞ are different and should be distinguished was made by Amit and Gross (1966). From (2.18) to (2.20) it follows that

$$\mathscr{E}_v - \mathscr{E}_u = \tfrac{1}{2}V_0 \int_{\mathscr{V}} (R^2 - R_u^2)^2 \, dV = \tfrac{1}{2}V_0 \int_{\mathscr{V}} (R^2 - R_\infty^2)^2 \, dV - \tfrac{1}{2}V_0 (R_\infty^2 - R_u^2)^2 \mathscr{V}. \tag{2.24}$$

Since, according to (2.23), $(R_\infty^2 - R_u^2)^2$ is $O(1/\mathscr{V}^2)$, in the limit $\mathscr{V} \to \infty$ ($\mathscr{R}$ fixed), we have, in the same limit,

$$\mathscr{E}_v - \mathscr{E}_u = \tfrac{1}{2}V_0 \int_{\mathscr{V}} (R^2 - R_\infty^2)^2 \, dV \tag{2.25}$$

i.e. in working out the increase in potential energy it is unnecessary, when form (2.25) is used, to make the distinction (2.23) between R_u and R_∞.

Outside the healing layers where the last term in (2.15) is small, the flow is governed essentially by Euler's equation for a barotropic fluid for which:

$$p = \left(\frac{E_v}{M\rho_\infty}\right)\rho^2 \tag{2.26}$$

giving by (2.10) Bernoulli's result for a compressible fluid, namely,

$$\frac{p}{\rho}+\tfrac{1}{2}(\nabla\phi)^2-\frac{\partial\phi}{\partial t}=\frac{p_\infty}{\rho_\infty} \tag{2.27}$$

where $p_\infty = E_v\rho_\infty/M$. In many situations of interest, E_v is large compared with $\frac{1}{2}Mu^2$ (or equivalently $\Phi \gg \frac{1}{2}u^2$). If in addition the flow is nearly steady ($|\partial \boldsymbol{u}/\partial t| \ll E_v/ML$), then (2.27) gives $\rho = \rho_\infty (= \text{constant})$, i.e. by (2.9),

$$\nabla^2\phi = 0 \tag{2.28}$$

which defines *incompressible* potential flow.

We now cast the basic equations in non-dimensional form by taking a as the unit of length, $\rho_\infty a^3$ as the unit of mass and $\hbar/2E_v$ as the unit of time, i.e.

$$\left.\begin{array}{lll} \boldsymbol{x}\to a\boldsymbol{x}, & t\to\dfrac{\hbar}{2E_v}t, & \boldsymbol{u}\to\dfrac{2aE_v}{\hbar}\boldsymbol{u} \\ \rho\to\rho_\infty\rho, & \phi\to\dfrac{\hbar}{M}\phi, & \Phi\to\dfrac{2E_v}{M}\Phi \end{array}\right\}. \tag{2.29}$$

Then equations (2.9) to (2.11) are unaltered, but (2.13) becomes

$$2\Phi = \rho-1-\frac{\nabla^2\rho^{1/2}}{\rho^{1/2}}. \tag{2.30}$$

In a steady flow, we have by (2.9) to (2.11),

$$\nabla.(\rho\nabla\phi) = 0 \tag{2.31}$$

$$\rho = 1-(\nabla\phi)^2+\frac{\nabla^2\rho^{1/2}}{\rho^{1/2}}. \tag{2.32}$$

In our non-dimensional units (2.29) writing

$$\mathscr{E}\to 2\rho_\infty a^3\frac{E_v}{M}\mathscr{E} \tag{2.33}$$

we obtain from (2.17)

$$\mathscr{E} = \tfrac{1}{2}\int_v (\nabla\rho^{1/2})^2\,\mathrm{d}V+\tfrac{1}{2}\int_v \rho(\nabla\phi)^2\,\mathrm{d}V+\tfrac{1}{4}\int_v \rho^2\,\mathrm{d}V. \tag{2.34}$$

We note that the variation δ of $\mathscr{E}$ subject to the constraint that $\int_v\rho\,\mathrm{d}V$ = constant is

$$\delta\mathscr{E} = \int_v [\delta\rho\{\tfrac{1}{2}(\nabla\phi)^2\}+\rho\nabla\phi\,.\,\nabla\delta\phi+(\nabla\rho^{1/2})\,.\,(\nabla\delta\rho^{1/2})+\tfrac{1}{2}\rho\delta\rho-\tfrac{1}{2}\delta\rho]\,\mathrm{d}V$$

(where a Lagrange multiplier of $\frac{1}{2}$ has been assumed so that the final results conform with the dimensionalization selected). Integrating by parts, assuming $\delta\rho$ and $\delta\phi$ vanish on the boundaries, we obtain

$$\delta\mathscr{E} = \int_v \left[\tfrac{1}{2}\delta\rho\left\{(\nabla\phi)^2+\rho-1-\frac{\nabla^2\rho^{1/2}}{\rho^{1/2}}\right\}-\delta\phi\{\nabla\,.\,(\rho\nabla\phi)\}\right]\mathrm{d}V. \tag{2.35}$$

If $\delta\mathscr{E}$ vanishes for all independent variations of $\delta\rho$ and $\delta\phi$, then (2.31) and (2.32) must hold. This variational principle for these equations is essentially the one Amit and

Gross (1966) used in their study of the circular vortex ring. Using (2.17), (2.20) and (2.25), the excess energy $\mathscr{E}-\mathscr{E}_0$ of the disturbed system over that of the corresponding uniform system is found, in dimensionless units, to be

$$\mathscr{E}-\mathscr{E}_0 = \int_{\mathscr{V}} \{\tfrac{1}{2}\rho(\nabla\phi)^2+\tfrac{1}{2}(\nabla\rho^{1/2})^2+\tfrac{1}{4}(1-\rho)^2\}\,\mathrm{d}V = \int_{\mathscr{V}} e_{\mathrm{D}}\,\mathrm{d}V \tag{2.36}$$

say.

A convenient alternative formulation of the equations (2.21) and (2.22) is through a vector potential (or stream function). Let

$$\boldsymbol{u} = \frac{1}{\rho}\nabla\times\boldsymbol{A} \tag{2.37}$$

where we assume

$$\nabla\,.\,\boldsymbol{A} = 0. \tag{2.38}$$

This automatically satisfies the continuity equation (2.31) and, from the irrotationality condition (curl $\boldsymbol{u} = 0$) implied by (2.11), we have

$$\nabla^2\boldsymbol{A} = \frac{1}{\rho}\nabla\rho\times(\nabla\times\boldsymbol{A}). \tag{2.39}$$

Also (2.32) becomes

$$\nabla^2\rho^{1/2} = \rho^{1/2}\left\{\rho-1+\frac{1}{\rho^2}(\nabla\times\boldsymbol{A})^2\right\}. \tag{2.40}$$

In particular, for the steady three-dimensional motion of a fluid with an axis of symmetry $0z$ we can introduce a stream function denoted by $\psi(=A_\theta/\tilde{\omega})$. If $(\tilde{\omega}, \theta, z)$ are cylindrical polar coordinates with z in the direction $0z$ then, setting $\rho = R^2$, we obtain

$$\boldsymbol{u} = \frac{1}{R^2}\mathrm{curl}\left(\frac{\psi}{\tilde{\omega}}\boldsymbol{i}_\theta\right) \tag{2.41}$$

where $\boldsymbol{i}_\theta$ is the unit vector in the θ direction. The basic equations (2.39) and (2.40) now become

$$\Delta\psi = \frac{2}{R}\nabla R\,.\,\nabla\psi \tag{2.42}$$

$$\nabla^2 R = R(R^2-1)+\frac{1}{R^3}(\nabla\psi)^2 \tag{2.43}$$

where

$$\Delta = \frac{\partial^2}{\partial\tilde{\omega}^2}-\frac{1}{\tilde{\omega}}\frac{\partial}{\partial\tilde{\omega}}+\frac{\partial^2}{\partial z^2} \tag{2.44}$$

$$\nabla^2 = \frac{\partial^2}{\partial\tilde{\omega}^2}+\frac{1}{\tilde{\omega}}\frac{\partial}{\partial\tilde{\omega}}+\frac{\partial^2}{\partial z^2}. \tag{2.45}$$

(The use of ψ in (2.41) to (2.45) and henceforward as a stream function, and not a wave function, should not cause confusion.)

3. Solution for a circular vortex

Measurements of the circulation of vortices in liquid He II have confirmed that, as predicted, the circulation is quantized in units of $\kappa = h/M$. In our non-dimensional units this quantum of circulation is 2π. We consider only a vortex ring with one unit of circulation, as these are energetically the most favourable; the ring has radius ac (or c in non-dimensional units) where $c \gg 1$. The frame of reference adopted is one in which the vortex is at rest and we can then apply the basic steady equations (2.42) and (2.43). The singularity in curl $\boldsymbol{u}$ is on the circle VV′, i.e. $z = 0$, $\tilde{\omega} = c$, and $0z$ is the axis of symmetry as shown in figure 1. We also introduce the coordinates (T, χ, θ),

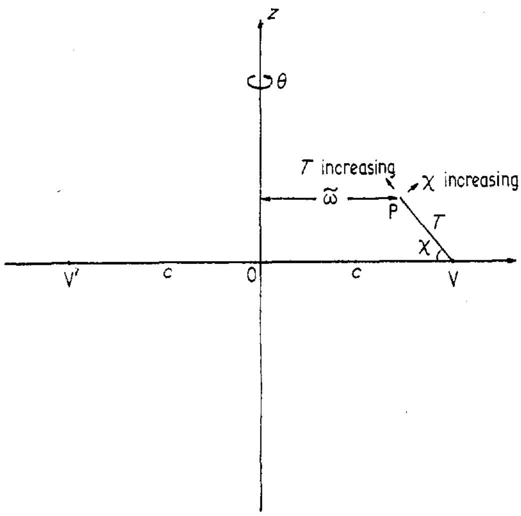

Figure 1. Illustrating the circular vortex and its related coordinate systems: $(\tilde{\omega}, \theta, z)$ are cylindrical coordinates, (T, χ, θ) are displaced polar coordinates. The singularity associated with the vortex line is indicated by V and V′ in the cross section shown.

often referred to as displaced polars, where

$$\tilde{\omega} = c - T\cos\chi$$
$$z = T\sin\chi.$$

Then the scale lengths and derivatives are

$$\left.\begin{aligned} h_T = 1, \qquad h_\chi = T, \qquad h_\theta = \tilde{\omega} \\ \frac{\partial}{\partial\tilde{\omega}} = -\cos\chi\frac{\partial}{\partial T}+\frac{\sin\chi}{T}\frac{\partial}{\partial\chi}, \qquad \frac{\partial}{\partial z} = \sin\chi\frac{\partial}{\partial T}+\frac{\cos\chi}{T}\frac{\partial}{\partial\chi}\end{aligned}\right\} \tag{3.1}$$

and (2.41) yields, in displaced polar components,

$$\boldsymbol{u} = \frac{1}{R^2(c-T\cos\chi)}\left(-\frac{1}{T}\frac{\partial\psi}{\partial\chi}, \frac{\partial\psi}{\partial T}, 0\right). \tag{3.2}$$

We now examine the structure of the solutions in the limit $c \to \infty$. The mathematical problem is one of inner and outer expansions, corresponding roughly to the solution 'inside' and 'outside' the core of the vortex where by 'core' we mean a toroid centered on VV′ whose radius T is, in physical units, O(a). For the inner expansion we iterate about the solution for the rectilinear vortex in the obvious way, and examine the form of the resulting solution in the limit of $T \to \infty$. In the outer region we use a stretched coordinate $s = T/c$ and examine the solution for $s \to 0$. Finally these two asymptotic solutions ($T \to \infty$ and $s \to 0$) are matched; the solution is then complete and quantities such as vortex energy and velocity can be computed.

Interior solution

Expanding ψ, R by

$$\psi = \sum_{n=0}^{\infty} c^{1-n}\psi_n(T,\chi), \qquad R = \sum_{n=0}^{\infty} c^{-n}R_n(T,\chi) \tag{3.3}$$

we see that as $c \to \infty$ with T fixed, we are always inside the 'core' of the vortex. Also as $c \to \infty$ the solution near to the singularity of curl $\boldsymbol{u}$ is increasingly like that for the straight vortex: thus ψ_0 and R_0 are the solutions for a straight vortex and both are independent of χ. When T and χ are used as independent variables, equations (2.42) and (2.43) become

$$\frac{\partial^2\psi}{\partial T^2}+\frac{1}{T^2}\frac{\partial^2\psi}{\partial\chi^2}+\frac{1}{T(c-T\cos\chi)}\left(c\frac{\partial\psi}{\partial T}-\sin\chi\frac{\partial\psi}{\partial\chi}\right) = \frac{2}{R}\left(\frac{\partial\psi}{\partial T}\frac{\partial R}{\partial T}+\frac{1}{T^2}\frac{\partial\psi}{\partial\chi}\frac{\partial R}{\partial\chi}\right) \tag{3.4}$$

and

$$\frac{\partial^2 R}{\partial T^2}+\frac{1}{T^2}\frac{\partial^2 R}{\partial\chi^2}+\frac{1}{T(c-T\cos\chi)}\left\{(c-2T\cos\chi)\frac{\partial R}{\partial T}+\sin\chi\frac{\partial R}{\partial\chi}\right\} = R(R^2-1)+\frac{1}{R^3(c-T\cos\chi)^2}\left\{\left(\frac{\partial\psi}{\partial T}\right)^2+\left(\frac{1}{T}\frac{\partial\psi}{\partial\chi}\right)^2\right\}. \tag{3.5}$$

On substituting (3.3) into (3.4) and equating coefficients of c and the constant term, we obtain

$$\frac{d^2\psi_0}{dT^2}+\frac{1}{T}\frac{d\psi_0}{dT} = \frac{2}{R_0}\frac{d\psi_0}{dT}\frac{dR_0}{dT} \tag{3.6}$$

$$\frac{\partial^2\psi_1}{\partial T^2}+\frac{1}{T^2}\frac{\partial^2\psi_1}{\partial\chi^2}+\cos\chi\frac{d\psi_0}{dT}+\frac{1}{T}\frac{\partial\psi_1}{\partial T} = \frac{2}{R_0}\left(\frac{\partial\psi_1}{\partial T}\frac{dR_0}{dT}-\frac{R_1}{R_0}\frac{d\psi_0}{dT}\frac{dR_0}{dT}+\frac{d\psi_0}{dT}\frac{\partial R_1}{\partial T}\right). \tag{3.7}$$

On substituting (3.3) into (3.5) and equating coefficients of the constant term and c^{-1}, we obtain

$$\frac{d^2R_0}{dT^2}+\frac{1}{T}\frac{dR_0}{dT} = R_0({R_0}^2-1)+\frac{1}{{R_0}^3}\left(\frac{d\psi_0}{dT}\right)^2 \tag{3.8}$$

and

$$\frac{\partial^2 R_1}{\partial T^2}+\frac{1}{T^2}\frac{\partial^2 R_1}{\partial\chi^2}+\frac{1}{T}\frac{\partial R_1}{\partial T}-\cos\chi\frac{dR_0}{dT} = 3R_1({R_0}^2-1)+\frac{2}{{R_0}^3}\frac{d\psi_0}{dT}\frac{\partial\psi_1}{\partial T}-\frac{3R_1}{{R_0}^4}\left(\frac{d\psi_0}{dT}\right)^2 +\frac{2T\cos\chi}{{R_0}^3}\left(\frac{d\psi_0}{dT}\right)^2. \tag{3.9}$$

Equation (3.6) can be integrated to give

$$\frac{d\psi_0}{dT} = \alpha\frac{{R_0}^2}{T} \qquad (\alpha = \text{constant}). \tag{3.10}$$

Now for $T \to 0$, $u_\chi \to d\psi_0/R_0^2 dT = \alpha/T$. By the condition on quantization of circulation, we must have

$$\int_0^{2\pi} u_\chi T \, d\chi = 2\pi. \tag{3.11}$$

Thus $\alpha = 1$ and by (3.10)

$$\frac{d\psi_0}{dT} = \frac{R_0^2}{T}. \tag{3.12}$$

After substituting for $d\psi_0/dT$ in (3.8), we obtain

$$\frac{d^2R_0}{dT^2} + \frac{1}{T}\frac{dR_0}{dT} = R_0\left(R_0^2 - 1 + \frac{1}{T^2}\right). \tag{3.13}$$

A solution to this equation is required in which R_0 goes asymptotically to 1 for $T \to \infty$ and $\psi_0 \to 0$ as $T \to 0$. For small T we have

$$R_0 = kT + O(T^3) \tag{3.14}$$

where k is a constant which can be obtained numerically, and for large T we have

$$R_0 = 1 - \frac{1}{2T^2} - \frac{9}{8T^4} + O\left(\frac{1}{T^6}\right). \tag{3.15}$$

Equation (3.13) was solved numerically subject to these boundary conditions. The solution is required later in evaluating numerical constants in the energy and the velocity of the vortex. The results are not presented here since they have already been given by Ginzburg and Pitaevskii (1958) and later, and in more detail, by Kawatra and Pathria (1966).

Integrating equation (3.12) we find that for large T

$$\psi_0 = K + \ln T + \frac{1}{2T^2} + \frac{1}{2T^4} + O\left(\frac{1}{T^6}\right) \tag{3.16}$$

where K is a constant. If we now write

$$\psi_1(\chi, T) = \cos\chi\,\psi_1(T), \qquad R_1(\chi, T) = \cos\chi\,R_1(T)$$

and substitute into (3.7) and (3.9) we obtain the fourth-order system

$$\frac{d^2\psi_1}{dT^2} + \frac{1}{T}\frac{d\psi_1}{dT} - \frac{\psi_1}{T^2} - \frac{2R_0}{T}\left(\frac{dR_1}{dT} - \frac{R_1}{R_0}\frac{dR_0}{dT}\right) - \frac{2}{R_0}\frac{dR_0}{dT}\frac{d\psi_1}{dT} = -\frac{R_0^2}{T} \tag{3.17}$$

$$\frac{d^2R_1}{dT^2} + \frac{1}{T}\frac{dR_1}{dT} + \frac{2R_1}{T^2} - (3R_0^2 - 1)R_1 - \frac{2}{R_0 T}\frac{d\psi_1}{dT} = \frac{2R_0}{T} + \frac{dR_0}{dT}. \tag{3.18}$$

We now show that the system must be subjected to four 'boundary' conditions, which therefore determine a unique solution.

For small T, the solutions for the complementary function defined by (3.17) and (3.18) are

$$\psi_1 \propto T^3, \qquad R_1 \propto T^2$$
$$\psi_1 \propto T \ln T, \qquad R_1 \propto \ln T$$
$$\psi_1 \propto T, \qquad R_1 \propto 1$$
$$\psi_1 \propto \frac{1}{T}, \qquad R_1 \propto \frac{1}{T^2}.$$

The second and fourth possibilities may be ruled out immediately on physical grounds. The third corresponds to an exact solution for the complementary function of the form $\psi_1 = \mathrm{d}\psi_0/\mathrm{d}T$, and $R_1 = \mathrm{d}R_0/\mathrm{d}T$. To include this would imply that the vortex core is at a point slightly displaced from $T = 0$, contrary to supposition. Thus, only the first solution is possible and, since a particular integral of the system is $\psi_1 = 0$, and $R_1 = \frac{1}{2}kT^2$, the general solution to (3.17) and (3.18) for $T \to 0$ is of the form

$$\psi_1 = \beta k^2 T^3, \qquad R_1 = \gamma k T^2 \tag{3.19}$$

where $\gamma - \beta = \frac{1}{2}$ and β(or γ) is as yet undetermined.

To examine the solution of (3.17) and (3.18) for large T, we substitute for R_0 from (3.15). The resulting equations are

$$\frac{\mathrm{d}^2\psi_1}{\mathrm{d}T^2}+\frac{1}{T}\frac{\mathrm{d}\psi_1}{\mathrm{d}T}-\frac{\psi_1}{T^2}-\left(\frac{2}{T^3}+\frac{10}{T^5}+\dots\right)\frac{\mathrm{d}\psi_1}{\mathrm{d}T}-\left(\frac{2}{T}-\frac{1}{T^3}+\dots\right)\left(\frac{\mathrm{d}R_1}{\mathrm{d}T}-\frac{R_1}{T^3}\right)$$
$$= -\frac{1}{T}+\frac{1}{T^3}+\frac{2}{T^5}+\dots \tag{3.20}$$

$$\frac{\mathrm{d}^2R_1}{\mathrm{d}T^2}+\frac{1}{T}\frac{\mathrm{d}R_1}{\mathrm{d}T}+\frac{5R_1}{T^2}-2R_1+\frac{6R_1}{T^4}-\left(\frac{2}{T}+\frac{1}{T^3}+\frac{11}{4T^5}+\dots\right)\frac{\mathrm{d}\psi_1}{\mathrm{d}T}$$
$$= \frac{2}{T}+\frac{9}{4T^5}+\dots . \tag{3.21}$$

The complete solutions are of the form

$$\psi_1 = -\tfrac{1}{2}T\ln T+BT+\frac{(\ln T)^2}{2T}-(4B+1)\frac{\ln T}{2T}+\mathrm{O}\left(\frac{1}{T}\right)$$
$$+D\exp(\sqrt{2}T)\left(\frac{\sqrt{2}}{T^{3/2}}+\frac{17}{8T^{5/2}}+\dots\right)+E\exp(-\sqrt{2}T)\left(-\frac{\sqrt{2}}{T^{3/2}}+\frac{17}{8T^{5/2}}+\dots\right) \tag{3.22}$$

$$R_1 = \frac{1}{2T}\ln T-\frac{(B+\frac{1}{2})}{T}+\frac{(\ln T)^2}{2T^3}+\frac{\ln T}{2T^3}(\tfrac{1}{2}-4B)+\frac{A}{T^3}-\frac{(6B+5)}{4T^3}+\dots$$
$$+D\exp(\sqrt{2}T)\left(\frac{1}{T^{1/2}}+\frac{5}{16}\sqrt{2}\frac{1}{T^{3/2}}+\dots\right)$$
$$+E\exp(-\sqrt{2}T)\left(\frac{1}{T^{1/2}}-\frac{5}{16}\sqrt{2}\frac{1}{T^{3/2}}+\dots\right) \tag{3.23}$$

where A, B, D and E are constants. The growing terms involving $\exp(\sqrt{2}T)$ cannot be matched with the outer solution to come, and we must therefore impose the fourth condition that $D = 0$, which essentially determines β. The problem is now clearly seen to be well posed. The terms involving A and B match with corresponding expressions in the expansion of the outer solution about $s = 0$.

Equations (3.17) and (3.18) have an interesting property which is important later. If we multiply (3.18) by $T(\mathrm{d}R_0/\mathrm{d}T)$, substitute into this for $\{(2\,\mathrm{d}R_0/\mathrm{d}T)\,.\,(\mathrm{d}\psi_0/\mathrm{d}T)\}/R_0$

from (3.17), and rearrange using (3.13) we obtain

$$\frac{\mathrm{d}}{\mathrm{d}T}\left\{T\frac{\mathrm{d}R_0}{\mathrm{d}T}\frac{\mathrm{d}R_1}{\mathrm{d}T} - TR_1\left(\frac{\mathrm{d}^2R_0}{\mathrm{d}T^2}\right) + \frac{2R_0R_1}{T} - \left(\frac{\mathrm{d}\psi_1}{\mathrm{d}T}+\frac{\psi_1}{T}\right)\right\}$$
$$= T\left(\frac{\mathrm{d}R_0}{\mathrm{d}T}\right)^2 + \frac{R_0^2}{T} + 2R_0\frac{\mathrm{d}R_0}{\mathrm{d}T}. \tag{3.24}$$

Integrating from 0 to ∞

$$\left(T\frac{\mathrm{d}R_0}{\mathrm{d}T}\frac{\mathrm{d}R_1}{\mathrm{d}T} - TR_1\frac{\mathrm{d}^2R_0}{\mathrm{d}T^2} + \frac{2R_1R_0}{T}\right)_0^\infty - \left(\frac{\mathrm{d}\psi_1}{\mathrm{d}T}+\frac{\psi_1}{T}\right)_0^\infty$$
$$= \int_0^\infty \left\{T\left(\frac{\mathrm{d}R_0}{\mathrm{d}T}\right)^2 + \frac{R_0^2}{T} + 2R_0\frac{\mathrm{d}R_0}{\mathrm{d}T}\right\}\mathrm{d}T. \tag{3.25}$$

Then, substituting the values of R_1, ψ_1, and R_0 for small and large T we have

$$\tfrac{1}{2} - 2B = \int_0^\infty T\left(\frac{\mathrm{d}R_0}{\mathrm{d}T}\right)^2 \mathrm{d}T + ⨍_0^\infty \frac{R_0^2}{T}\mathrm{d}T + 1 \tag{3.26}$$

where B is the constant appearing in (3.22) and (3.23). Again the bar through the second integral denotes its finite part, for although R_0^2/T behaves like $1/T$ for large T, the divergent logarithmic term is cancelled by a similar term from the left-hand side of (3.25). Thus

$$-2B = \int_0^\infty T\left(\frac{\mathrm{d}R_0}{\mathrm{d}T}\right)^2 \mathrm{d}T + ⨍_0^\infty \frac{R_0^2}{T}\mathrm{d}T + \tfrac{1}{2}. \tag{3.27}$$

Exterior solution

For the outer variable we replace T by $s = T/c$, so that as $c \to \infty$ with s fixed, we have the solution 'outside the core' of the vortex. That is we should approximate to a region of constant density in which ψ will give a flow common to the exterior of all vortex rings. With respect to s and χ, equations (3.4) and (3.5) become

$$\frac{\partial^2\psi}{\partial s^2} + \frac{1}{s^2}\frac{\partial^2\psi}{\partial\chi^2} + \frac{1}{s(1-s\cos\chi)}\left(\frac{\partial\psi}{\partial s} - \sin\chi\frac{\partial\psi}{\partial\chi}\right) = \frac{2}{R}\left(\frac{\partial\psi}{\partial s}\frac{\partial R}{\partial s} + \frac{1}{s^2}\frac{\partial\psi}{\partial\chi}\frac{\partial R}{\partial\chi}\right) \tag{3.28}$$

$$\frac{\partial^2 R}{\partial s^2} + \frac{1}{s^2}\frac{\partial^2 R}{\partial\chi^2} + \frac{1}{s(1-s\cos\chi)}\left\{(1-2s\cos\chi)\frac{\partial R}{\partial s} + \sin\chi\frac{\partial R}{\partial\chi}\right\}$$
$$= c^2R(R^2-1) + \frac{1}{c^2R^3}\frac{1}{(1-s\cos\chi)^2}\left\{\left(\frac{\partial\psi}{\partial s}\right)^2 + \frac{1}{s^2}\left(\frac{\partial\psi}{\partial\chi}\right)^2\right\}. \tag{3.29}$$

Expanding ψ and R by

$$\psi = \sum_{n=0}^{\infty} c^{1-2n}\psi_n(s,\chi), \qquad R = \sum_{n=0}^{\infty} c^{-2n}R_n(s,\chi) \tag{3.30}$$

substituting (3.30) into (3.29) and equating powers of c^2 we obtain

$$R_0(R_0^2-1) = 0$$

and hence $R_0 = 1$. The constant terms in (3.29) then give

$$R_1 = -\frac{1}{2}\frac{1}{(1-s\cos\chi)^2}\left\{\left(\frac{\partial\psi_0}{\partial s}\right)^2 + \frac{1}{s^2}\left(\frac{\partial\psi_0}{\partial\chi}\right)^2\right\}. \tag{3.31}$$

Substituting (3.30) into (3.28), we obtain

$$\Delta\psi_0 = 0 \tag{3.32}$$

where the operator is given by (2.44), and

$$\frac{\partial^2\psi_1}{\partial s^2}+\frac{1}{s^2}\frac{\partial^2\psi_1}{\partial\chi^2}+\frac{1}{s(1-s\cos\chi)}\left(\frac{\partial\psi_1}{\partial s}-\sin\chi\frac{\partial\psi_1}{\partial\chi}\right) = 2\left(\frac{\partial\psi_0}{\partial s}\frac{\partial R_1}{\partial s}+\frac{1}{s^2}\frac{\partial\psi_0}{\partial\chi}\frac{\partial R_1}{\partial\chi}\right). \tag{3.33}$$

From (3.29) we also have

$$2R_2 = -9R_1{}^2+\frac{\partial^2 R_1}{\partial s^2}+\frac{1}{s^2}\frac{\partial^2 R_1}{\partial\chi^2}+\frac{1}{s(1-s\cos\chi)}\left\{(1-2s\cos\chi)\frac{\partial R_1}{\partial s}+\sin\chi\frac{\partial R_1}{\partial\chi}\right\}$$
$$-\frac{2}{(1-s\cos\chi)^2}\left(\frac{\partial\psi_0}{\partial s}\frac{\partial\psi_1}{\partial s}+\frac{1}{s^2}\frac{\partial\psi_0}{\partial\chi}\frac{\partial\psi_1}{\partial\chi}\right). \tag{3.34}$$

The appropriate solutions of $\Delta\psi = 0$ for a ring are given by Dyson (1893) as linear combinations of the elementary solutions $\psi = \tilde{\omega}\mathrm{J}_n(\tilde{\omega}, z)$, where

$$\tilde{\omega}\mathrm{J}_{p+1}(\tilde{\omega}, z) = \tilde{\omega}\left(-\frac{1}{c}\frac{\mathrm{d}}{\mathrm{d}c}\right)^p\int_0^\pi \frac{\cos\phi\,\mathrm{d}\phi}{(z^2+c^2-2c\tilde{\omega}\cos\phi+\tilde{\omega}^2)^{1/2}}.$$

Dyson also provides expansions of these solutions about $s = 0$; these are

$$\tilde{\omega}\mathrm{J}_1 = l-\tfrac{1}{2}\,ls\cos\chi-\tfrac{1}{2}\,s\cos\chi-\tfrac{1}{16}\,ls^2\cos 2\chi+\tfrac{1}{8}ls^2+\ldots \tag{3.35}$$

$$\tilde{\omega}\mathrm{J}_2 = \frac{1}{c^2s}(\cos\chi+\tfrac{1}{2}ls+\ldots) \tag{3.36}$$

etc., where

$$l = \ln(\tfrac{8}{s})-2.$$

In the present case, to the zeroth order, we have

$$\psi_0 = \frac{1}{2c}U_0\tilde{\omega}^2+A_1\tilde{\omega}\mathrm{J}_1 \tag{3.37}$$

where A_1 is a constant to be determined by matching the inner and outer solutions. The growing solution $U_0\tilde{\omega}^2/2c$ corresponds to a uniform flow at infinity. In our frame the vortex is at rest and the fluid at infinity is moving. In a frame in which the fluid at infinity is at rest, U_0 will represent the speed of the vortex ring. A term like this might have to be included at any level in the expansion for ψ. The U_0 here can only give the translational velocity of the ring to the leading order in c^{-2}. Substituting ψ_0 into equation (3.31) we obtain

$$R_1 = -\frac{A_1{}^2}{2s^2}-\frac{A_1{}^2 l\cos\chi}{2s}+A_1\frac{\cos\chi}{s}(-U_0c-A_1)+\mathrm{O}(l) \tag{3.38}$$

and (3.33) then becomes

$$\frac{\partial^2\psi_1}{\partial s^2}+\frac{1}{s^2}\frac{\partial^2\psi_1}{\partial\chi^2}+\frac{1}{s(1-s\cos\chi)}\left(\frac{\partial\psi_1}{\partial s}-\sin\chi\frac{\partial\psi_1}{\partial\chi}\right)$$
$$= 2\left(-\frac{A_1{}^3}{s^4}-\frac{A_1{}^3 l\cos\chi}{s^3}-\frac{2A_1{}^2U_0c\cos\chi}{s^3}-\frac{3A_1{}^3\cos\chi}{2s^3}+\ldots\right). \tag{3.39}$$

This equation gives

$$\psi_1 = -\frac{1}{2c}U_1\tilde{\omega}^2 + c^2A_2\tilde{\omega}J_2 - \frac{A_1^3}{2s^2} - \frac{A_1^3}{2s}l^2\cos\chi + \frac{A_1^2}{s}\left(-2U_0c - \frac{3A_1}{2}\right)l\cos\chi + \dots \tag{3.40}$$

where A_2 is a constant and equation (3.34) for R_2 yields

$$R_2 = -\frac{A_1^2}{8s^4}(A_1^2+8) + A_1^4\frac{l^2}{2s^3}\cos\chi + (2U_0c + \tfrac{3}{4}A_1)A_1^3\frac{l}{s^3}\cos\chi + O\left(\frac{1}{s^3}\right). \tag{3.41}$$

Matching

The inner solution for $T \to \infty$ must match the outer solution for $s \to 0$ to all terms. The process fixes all the unknown constants and gives a consistent solution. Letting $\lambda = \ln 8c - 2$, then $\ln T = \ln sc = \lambda - l$. From the inner solution for R for large T, we have

$$R = \left(1 - \frac{1}{2T^2} - \frac{9}{8T^4} + \dots\right) + \frac{\cos\chi}{c}\left\{\frac{\ln T}{2T} - \frac{(B+\frac{1}{2})}{T} + \frac{(\ln T)^2}{2T^3} + \frac{\ln T}{2T^3}(\tfrac{1}{2} - 4B) + \frac{A}{T^3} - \frac{(6B+5)}{T^3} + \dots + E\exp(-\sqrt{2}T)\left(\frac{1}{T^{1/2}} + \dots\right)\right\}.$$

When T is replaced by sc and $\ln T$ by $\lambda - l$, discarding the experimentally small term, we obtain

$$R = 1 + \frac{1}{c^2}\left\{-\frac{1}{2s^2} - \frac{l}{2s}\cos\chi + \left(\frac{\lambda}{2} - B - \frac{1}{2}\right)\frac{1}{s}\cos\chi + \dots\right\} + \frac{1}{c^4}\left\{-\frac{9}{8s^4} + \frac{l^2}{2s^3}\cos\chi + \frac{l}{s^3}\cos\chi(-\lambda - \tfrac{1}{4} + 2B) + \dots\right\}. \tag{3.42}$$

From the outer solution for R for small s we have

$$R = 1 + \frac{1}{c^2}\left\{-\frac{A_1^2}{2s^2} - A_1^2\frac{l}{2s}\cos\chi - A_1\frac{\cos\chi}{s}(U_0c + A_1) + \dots\right\} + \frac{1}{c^4}\left\{-\frac{A_1^2}{8s^4}(A_1^2+8) + A_1^2\frac{l^2}{2s^3}\cos\chi + A_1^3\frac{l}{s^3}\cos\chi(2U_0c + \tfrac{3}{4}A_1) + \dots\right\}. \tag{3.43}$$

A comparison of corresponding terms in (3.42) and (3.43) yields

$$A_1^2 = 1, \qquad -A_1(U_0c + A_1) = \tfrac{1}{2}\lambda - B - \tfrac{1}{2}, \qquad A_1^3(2U_0c + \tfrac{3}{4}A_1) = -\lambda + 2B - \tfrac{1}{4}. \tag{3.44}$$

Similarly by matching the inner and outer solutions for ψ we obtain the relations

$$A_1 = -1, \qquad -(U_0c + \tfrac{1}{2}A_1) = B - \tfrac{1}{2}\lambda, \qquad A_1^3 = -1, \qquad -A_1^2(2U_0c + \tfrac{3}{2}A_1) = \tfrac{1}{2}(-2\lambda + 4B + 1). \tag{3.45}$$

All of these equations (3.44) and (3.45) show that

$$A_1 = -1, \qquad U_0c = \tfrac{1}{2}(\lambda + 1 - 2B). \tag{3.46}$$

Substituting for $-2B$ from (3.27) we finally obtain

$$U_0 = \frac{1}{2c}\left[\ln 8c - 1 + \left\{ \int_0^\infty \frac{R_0^2}{T}\,\mathrm{d}T + \int_0^\infty T\left(\frac{\mathrm{d}R_0}{\mathrm{d}T}\right)^2 \mathrm{d}T + \tfrac{1}{2}\right\}\right]. \tag{3.47}$$

It is worth noting that the relations (3.46) are confirmed if the inner expansion is taken to order c^{-3} and matched with the outer solution, although it is necessary to proceed even further if U_0 and $\mathscr{E}$ are to be obtained to a higher order.

4. Energy and impulse

The excess energy of the system has already been given in (2.36). In the present application to a circular vortex, this may be considered to be composed of a core energy, $\mathscr{E}_\mathrm{I}$ (say), and a contribution $\mathscr{E}_\mathrm{E}$ from outside the core. Since, however, the quantum vortex (unlike its classical counterpart) does not have a precisely defined core surface, this division is to some extent arbitrary. To be more definite, we introduce an arbitrary function $q(c)$ such that $1 \ll q \ll c$, for $c \to \infty$. Then the distance $T = q$ goes to ∞ with c (since $q \gg 1$) while in the same limit $s = q/c$ tends to zero (since $q \ll c$). Thus $T = \mathrm{O}(q)$ corresponds to the matching region used at the end of § 3, which may itself be regarded as bounding the core of the vortex.

The integral (2.36) is divided into

$$\mathscr{E}_\mathrm{I} = \int_{T<q} e_\mathrm{D}\,\mathrm{d}V \tag{4.1}$$

in which e_D is evaluated using the inner solution, and

$$\mathscr{E}_\mathrm{E} = \int_{T>q} e_\mathrm{D}\,\mathrm{d}V \tag{4.2}$$

in which e_D is evaluated using the outer solution. The former integral diverges as $q \to \infty$, and the latter integral diverges as $q/c \to 0$, but, when added to give $\mathscr{E} - \mathscr{E}_0$, the divergences cancel as we shall see. This is only to be expected.

In computing the kinetic energy, and impulse of the vortex flow, we must return to the coordinate frame in which the fluid velocity is zero at infinity. This is achieved by subtracting from ψ_0 in (3.37) the uniform flow. In the inner region

$$\psi = c\psi_0 + \psi_1 + \dots, \qquad R = R_0 + \frac{1}{c}R_1 + \dots, \qquad \mathrm{d}V = T(c - T\cos\chi)\,\mathrm{d}\chi\,\mathrm{d}T\,\mathrm{d}\theta.$$

Then using (4.1) we find that to the leading order in c

$$\mathscr{E}_\mathrm{I} = \tfrac{1}{2}\int_0^q \mathrm{d}T \int_0^{2\pi} \mathrm{d}\chi \int_0^{2\pi} cT\left\{\frac{1}{R_0^2}\left(\frac{\mathrm{d}\psi_0}{\mathrm{d}T}\right)^2 + \left(\frac{\mathrm{d}R_0}{\mathrm{d}T}\right)^2 + \tfrac{1}{2}(1-R_0^2)^2\right\}\mathrm{d}\theta$$

or using (3.12)

$$\mathscr{E}_\mathrm{I} = 2\pi^2 c\int_0^q \left\{\frac{R_0^2}{T} + T\left(\frac{\mathrm{d}R_0}{\mathrm{d}T}\right)^2 + \frac{T}{2}(1-R_0^2)^2\right\}\mathrm{d}T \tag{4.3}$$

which as $q \to \infty$ can be rewritten as (see statement preceding (3.27))

$$\mathscr{E}_\mathrm{I} = 2\pi^2 c\left\{\ln q + \int_0^q \frac{R_0^2}{T}\,\mathrm{d}T + \int_0^q T\left(\frac{\mathrm{d}R_0}{\mathrm{d}T}\right)^2 \mathrm{d}T + \tfrac{1}{2}\int_0^q (1-R_0^2)^2\,\mathrm{d}T\right\}. \tag{4.4}$$

Also in connection with the remark made after (4.2) we further rewrite this as

$$\mathscr{E}_{\mathrm{I}} = 2\pi^2 c\left\{(\ln 8c-2)-\left(\ln\frac{8c}{q}-2\right)+ \fint_0^q \frac{R_0^2}{T}\,\mathrm{d}T+\int_0^q T\left(\frac{\mathrm{d}R_0}{\mathrm{d}T}\right)^2\mathrm{d}T + \tfrac{1}{2}\int_0^q (1-R_0^2)^2 T\,\mathrm{d}T\right\}. \tag{4.5}$$

In the outer region

$$R = 1+\frac{R_1}{c^2}+\ldots$$

so that to the leading order

$$\mathscr{E}_{\mathrm{E}} = \tfrac{1}{2}\int_{T>q} (\nabla\phi)^2\,\mathrm{d}V. \tag{4.6}$$

Outside the core we have from (2.28)

$$\nabla^2\phi = 0. \tag{4.7}$$

If we now apply the divergence theorem to (4.6) in the simply connected region outside a very thin disk S_{D} which encloses the vortex 'core' we obtain

$$\mathscr{E}_{\mathrm{E}} = \tfrac{1}{2}\int_{S_{\mathrm{D}}} \phi\nabla\phi\,.\,\mathrm{d}\boldsymbol{S}+\tfrac{1}{2}\int_{S_\infty}\phi\nabla\phi\,.\,\mathrm{d}\boldsymbol{S} \tag{4.8}$$

where S_∞ is the sphere at infinity of radius R. The contribution to the integral from S_∞ is of order R^{-3} and can be neglected. We are left with

$$\mathscr{E}_{\mathrm{E}} = \tfrac{1}{2}\int_{S_{\mathrm{D}}} \phi\nabla\phi\,.\,\mathrm{d}\boldsymbol{S}. \tag{4.9}$$

The velocity potential at any point P due to a vortex is given by

$$\phi_{\mathrm{P}} = \frac{\kappa}{4\pi}\Omega_{\mathrm{P}}$$

where Ω_{P} is the solid angle subtended by P at the singularity of the vortex (Lamb 1945). Then if S_{D^+} and S_{D^-} are the upper and lower sides of S_{D}, we have

$$\phi_{S_{\mathrm{D}^+}} = \frac{\kappa}{2} = \pi, \qquad \phi_{S_{\mathrm{D}^-}} = -\frac{\kappa}{2} = -\pi. \tag{4.10}$$

Also $\nabla\phi\,.\,\mathrm{d}\boldsymbol{S} = (\partial\psi/\tilde{\omega}\,\partial\tilde{\omega})2\pi\tilde{\omega}\,\mathrm{d}\tilde{\omega}$, and hence, to the leading order, we obtain

$$\mathscr{E}_{\mathrm{E}} = 2\pi^2 c\left\{\ln\left(\frac{8c}{q}\right)-2\right\}. \tag{4.11}$$

Adding together $\mathscr{E}_{\mathrm{I}}$ and $\mathscr{E}_{\mathrm{E}}$ from (4.5) and (4.11) as $c\to\infty$ ($q\to\infty$), we find

$$\mathscr{E}-\mathscr{E}_0 = 2\pi c^2\left\{\ln(8c-2)+\fint_0^\infty \frac{R_0^2}{T}\,\mathrm{d}T+\int_0^\infty T\left(\frac{\mathrm{d}R_0}{\mathrm{d}T}\right)^2\mathrm{d}T+\tfrac{1}{2}\int_0^\infty T(1-R_0^2)^2\,\mathrm{d}T\right\}. \tag{4.12}$$

The last integral in (4.12) can be evaluated by integrating by parts, and using (3.13), (3.14) and (3.15), yielding the value $\frac{1}{2}$. We therefore finally obtain

$$\mathscr{E}-\mathscr{E}_0 = 2\pi c^2\left[\ln(8c-2)+\left\{\int_0^\infty \frac{R_0{}^2}{T}\,\mathrm{d}T+\int_0^\infty T\left(\frac{\mathrm{d}R_0}{\mathrm{d}T}\right)^2\mathrm{d}T+\tfrac{1}{2}\right\}\right]. \qquad (4.13)$$

It is easily shown that when the formal definition, that is,

$$\boldsymbol{p} = \int_{\mathscr{V}} \rho \boldsymbol{u}\,\mathrm{d}V$$

is used to calculate the momentum of a classical vortex, improper integrals are obtained which can be made to take many values depending, in the limit as the volume of integration $\mathscr{V}\to\infty$, upon the shape of that volume (Lin 1963). The ambiguity can be overcome by introducing the concept of impulse. It should be emphasized that, even in the quantum case, the formal definition of momentum based on (1.9) leads to improper integrals by analogy with the above. Impulse is, then, *not* a classical notion without a quantum counterpart. In fact it appears that the correct impulse p for the quantum system is precisely that which is obtained to leading order for the classical system, that is,

$$p = \pi\kappa\rho_\infty c^2.$$

In our non-dimensional variables this becomes

$$p = 2\pi^2 c^2. \qquad (4.14)$$

Collecting together the results of (3.47), (4.13) and (4.14) we have

$$\left.\begin{aligned} U_0 &= \frac{1}{2c}\left[\ln(8c-1)+\left\{\int_0^\infty \frac{R_0{}^2}{T}\,\mathrm{d}T+\int_0^\infty T\left(\frac{\mathrm{d}R_0}{\mathrm{d}T}\right)^2\mathrm{d}T+\tfrac{1}{2}\right\}\right]\\ \mathscr{E}-\mathscr{E}_0 &= 2\pi^2 c\left[\ln(8c-2)+\left\{\int_0^\infty \frac{R_0{}^2}{T}\,\mathrm{d}T+\int_0^\infty T\left(\frac{\mathrm{d}R_0}{\mathrm{d}T}\right)^2\mathrm{d}T+\tfrac{1}{2}\right\}\right]\\ p &= 2\pi^2c^2 \end{aligned}\right\} \qquad (4.15)$$

and it is easily seen that these quantities obey the Hamiltonian relation

$$U_0 = \frac{\partial\mathscr{E}}{\partial p}.$$

The constant terms in brackets in (4.15) were evaluated by a numerical integration of (3.13); the values obtained agreed with those obtained by Ginzburg and Pitaevskii (1958). We found, in physical variables,

$$\left.\begin{aligned} U_0 &= \frac{\kappa}{4\pi c}\left\{\ln\left(\frac{8c}{a}\right)-0{\cdot}615\right\}\\ \mathscr{E}-\mathscr{E}_0 &= \tfrac{1}{2}\rho\kappa^2 c\left\{\ln\left(\frac{8c}{a}\right)-1{\cdot}615\right\} \end{aligned}\right\}. \qquad (4.16)$$

It should be emphasized that these results are to be regarded as providing the leading pair of terms in the asymptotic expansions of these quantities as $c/a\to\infty$ (the next terms would involve $(a/c)^2\ln(8c/a)$ in the braces), and as such they are (unlike the results of Amit and Gross 1966) in principle exact (the only uncertainties arising from the numerical quadrature).

In the sequel to this paper, one of us (J.G.) will consider the oscillation of these vortex lines. Some of the results given in this paper were presented by one of us (P.H.R.) at the British Applied Mechanics Congress, held in Nottingham at Easter, 1969.

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